that these conditions are also necessary. It remains, therefore, only to examine the case when H has two independent generators whose orders are not powers of the same prime number.

In this general case, we may write h in the form $3^{a_0} p_1^{a_1} p_2^{a_2} \ldots$ and observe that each of the subgroups whose orders are $3^{a_0}, p_1^{a_1}, p_2^{a_2}, \ldots$ is either cyclic or the direct product of two cyclic groups.* The necessary and sufficient conditions that s_1 and s_2 can be so chosen as to generate a group which contains any one of these subgroups, as H, have been determined. If these conditions are satisfied for each of the given subgroups, we may suppose s_1 and s_2 so formed as to generate the group formed by establishing a $(3^{a_0}, p_1^{a_1}, p_2^{a_2} \ldots)$ isomorphism between the separate group whose h's are powers of primes. If they are not satisfied in each instance, it follows from the equation $s_3 s_4 s_5 = 1$ that there is no group in the infinite system under consideration which has this H. Hence this general case is included under the special cases considered above.

CORNELL UNIVERSITY, JULY, 1901.

ON THE INVARIANTS OF A QUADRANGLE UNDER THE LARGEST SUBGROUP, HAVING A FIXED POINT, OF THE GENERAL PROJECTIVE GROUP IN THE PLANE.**

By W. A. GRANVILLE.

IN the ANNALS OF MATHEMATICS, vol. 12, p. 82, Professor Lovett proposes the problem of finding the invariants of a quadrangle under the transformations of the six parameter group in the plane generated by the infinitesimal transformations:

$$\boxed{xp \quad yp \quad xq \quad yq \quad x^2p + xyq \quad xyp + y^2q},$$

$$p = \frac{\partial f}{dx}, \qquad q = \frac{\partial f}{dy}.$$

This is the largest subgroup of the general projective group in the plane, which has a fixed point. The invariants found by Professor Lovett were

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* *Cf. Bull. Amer. Math. Soc.*, vol. 7, 1901, p. 424.

** Presented to the American Mathematical Society at its meeting, April 27, 1901.

$$c_1 = \frac{y_3}{y_4} \cdot \frac{x_4 y_2 - x_2 y_4}{x_3 y_2 - x_2 y_3}, \qquad c_2 = \frac{x_2}{x_3} \cdot \frac{x_1 y_3 - x_3 y_1}{x_1 y_2 - x_2 y_1}.$$

On trial it will be found that these functions do not satisfy the differential equations of the complete system (6) on page 82:

$$\begin{aligned} &\sum_{i=1}^{4} x_i \frac{\partial \phi}{\partial x_i} = \sum_{i=1}^{4} y_i \frac{\partial \phi}{\partial x_i} = \sum_{i=1}^{4} x_i \frac{\partial \phi}{\partial y_i} = \sum_{i=1}^{4} y_i \frac{\partial \phi}{\partial y_i} = 0, \\ &\sum_{i=1}^{4} \left\{ x_i^2 \frac{\partial \phi}{\partial x_i} + x_i y_i \frac{\partial \phi}{\partial y_i} \right\} = \sum_{i=1}^{4} \left\{ x_i y_i \frac{\partial \phi}{\partial x_i} + y_i^2 \frac{\partial \phi}{\partial y_i} \right\} = 0; \end{aligned} \qquad (6)$$

and hence cannot be the invariant functions.

The two solutions of (6) are found to be

$$c_1 = \frac{\left(\frac{y_1}{x_1} - \frac{y_3}{x_3}\right)\left(\frac{y_2}{x_2} - \frac{y_4}{x_4}\right)}{\left(\frac{y_1}{x_1} - \frac{y_4}{x_4}\right)\left(\frac{y_2}{x_2} - \frac{y_3}{x_3}\right)} = \frac{m_{01} - m_{03}}{m_{01} - m_{04}} \cdot \frac{m_{02} - m_{04}}{m_{02} - m_{03}};$$

and

$$c_2 = \frac{\left(\frac{y_1}{x_1} - \frac{y_1 - y_3}{x_1 - x_3}\right)\left(\frac{y_1 - y_4}{x_1 - x_4} - \frac{y_1 - y_2}{x_1 - x_2}\right)}{\left(\frac{y_1}{x_1} - \frac{y_1 - y_4}{x_1 - x_4}\right)\left(\frac{y_1 - y_3}{x_1 - x_3} - \frac{y_1 - y_2}{x_1 - x_2}\right)} = \frac{m_{10} - m_{13}}{m_{10} - m_{14}} \cdot \frac{m_{14} - m_{12}}{m_{13} - m_{12}};$$

where m_{ik} is the slope of the line drawn from the point (x_i, y_i) to the point (x_k, y_k), the origin being denoted by (x_0, y_0). Hence the theorem:

If a quadrangle (1234) *be transformed by the Lie group*

$$\boxed{xp \quad yp \quad xq \quad yq \quad x^2p + xyq \quad xyp + y^2q},$$

the cross ratio of the pencil of lines drawn from any vertex of the pentagon (01234) *to the remaining four vertices remains constant.*

If the problem be considered from the standpoint of projective geometry, this result follows at once from the well known fact that the cross ratio of a pencil of any four concurrent lines is an invariant under any projective transformation. It is also evident that two of the cross ratios here considered are independent.

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